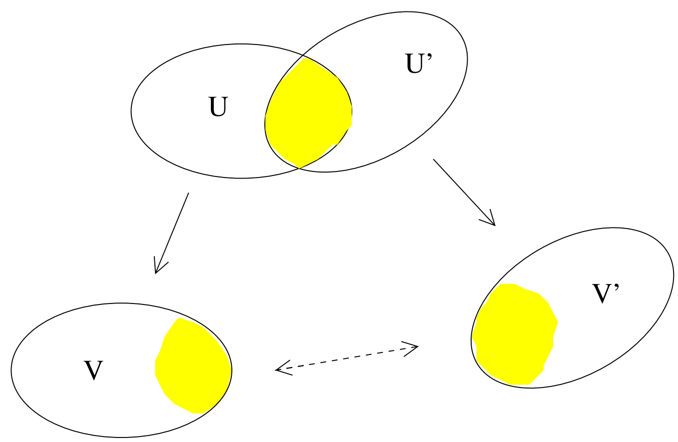
3 Riemann surfaces

3.1 Definitions and examples

From the definition of a surface, each point has a neighbourhood U and a homeomorphism φ_U from U to an open set V in $\mathbf{R}^2$. If two such neighbourhoods U, U' intersect, then

$$\varphi_{U'}\varphi_U^{-1} : \varphi_U(U \cap U') \to \varphi_{U'}(U \cap U')$$

is a homeomorphism from one open set of $\mathbf{R}^2$ to another.



If we identify $\mathbf{R}^2$ with the complex numbers $\mathbf{C}$ then we can define:

Definition 8 *A Riemann surface is a surface with a class of homeomorphisms φ_U such that each map $\varphi_{U'}\varphi_U^{-1}$ is a* holomorphic *(or* analytic*) homeomorphism.*

We call each function φ_U a holomorphic coordinate.

Examples:

1. Let X be the extended complex plane $X = \mathbf{C} \cup \{\infty\}$. Let $U = \mathbf{C}$ with $\varphi_U(z) = z \in \mathbf{C}$. Now take

$$U' = \mathbf{C}\backslash\{0\} \cup \{\infty\}$$

and define $z' = \varphi_{U'}(z) = z^{-1} \in \mathbf{C}$ if $z \neq \infty$ and $\varphi_{U'}(\infty) = 0$. Then

$$\varphi_U(U \cap U') = \mathbf{C}\backslash\{0\}$$

and

$$\varphi_U\varphi_{U'}^{-1}(z) = z^{-1}$$

which is holomorphic.

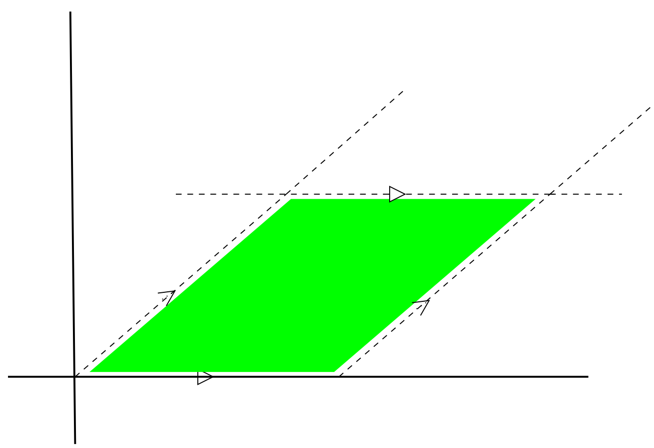
In the right coordinates this is the sphere, with ∞ the North Pole and the coordinate maps given by stereographic projection. For this reason it is sometimes called the *Riemann sphere*.

2. Let $\omega_1, \omega_2 \in \mathbf{C}$ be two complex numbers which are linearly independent over the reals, and define an equivalence relation on $\mathbf{C}$ by $z_1 \sim z_2$ if there are integers m, n such that $z_1 - z_2 = m\omega_1 + n\omega_2$. Let X be the set of equivalence classes (with the quotient topology). A small enough disc V around $z \in \mathbf{C}$ has at most one representative in each equivalence class, so this gives a local homeomorphism to its projection U in X. If U and U' intersect, then the two coordinates are related by a map

$$z \mapsto z + m\omega_1 + n\omega_2$$

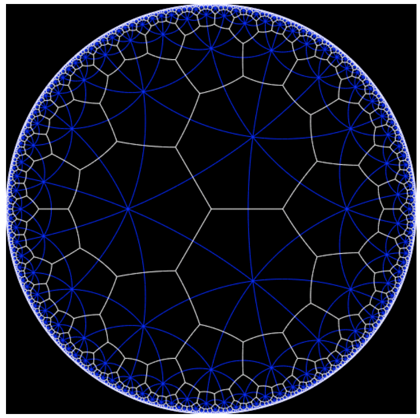
which is holomorphic.

This surface is topologically described by noting that every z is equivalent to one inside the closed parallelogram whose vertices are $0, \omega_1, \omega_2, \omega_1 + \omega_2$, but that points on the boundary are identified:



We thus get a torus this way. Another way of describing the points of the torus is as *orbits* of the action of the group $\mathbf{Z} \times \mathbf{Z}$ on $\mathbf{C}$ by $(m, n) \cdot z = z + m\omega_1 + n\omega_2$.

3. The parallelograms in Example 2 fit together to tile the plane. There are groups of holomorphic maps of the unit disc into itself for which the interior of a polygon plays the same role as the interior of the parallelogram in the plane, and we get a surface X by taking the orbits of the group action. Now we get a tiling of the disc:



In this example the polygon has eight sides and the surface is homeomorphic by the classification theorem to the connected sum of two tori.

4. A *complex algebraic curve* X in $\mathbf{C}^2$ is given by

$$X = \{(z, w) \in \mathbf{C}^2 : f(z, w) = 0\}$$

where f is a polynomial in two variables with complex coefficients. If $(\partial f/\partial z)(z, w) \neq 0$ or $(\partial f/\partial w)(z, w) \neq 0$ for every $(z, w) \in X$, then using the implicit function theorem (see Appendix A) X can be shown to be a Riemann surface with local homeomorphisms given by

$$(z, w) \mapsto w \text{ where } (\partial f/\partial z)(z, w) \neq 0$$

and

$$(z, w) \mapsto z \text{ where } (\partial f/\partial w)(z, w) \neq 0.$$

Definition 9 *A holomorphic map between Riemann surfaces X and Y is a continuous map $f : X \to Y$ such that for each holomorphic coordinate φ_U on U containing x on X and ψ_W defined in a neighbourhood of $f(x)$ on Y, the composition*

$$\psi_W \circ f \circ \varphi_U^{-1}$$

is holomorphic.

In particular if we take $Y = \mathbf{C}$, we can define holomorphc functions on X.

Before proceeding, recall some basic facts about holomorphic functions (see [4]):

- A holomorphic function has a convergent power series expansion in a neighbourhood of each point at which it is defined:

$$f(z) = a_0 + a_1(z-c) + a_2(z-c)^2 + \dots$$

- If f vanishes at c then

$$f(z) = (z-c)^m(c_0 + c_1(z-c) + \dots)$$

  where $c_0 \neq 0$. In particular zeros are isolated.

- If f is non-constant it maps open sets to open sets.

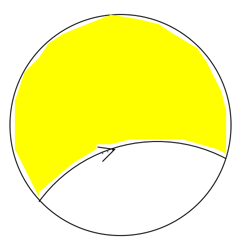
- $|f|$ cannot attain a maximum at an interior point of a disc ("maximum modulus principle").

- $f : \mathbf{C} \mapsto \mathbf{C}$ preserves angles between differentiable curves, both in magnitude and sense.

This last property shows:

Proposition 3.1 *A Riemann surface is orientable.*

Proof: Assume X contains a Möbius band, and take a smooth curve down the centre: $\gamma : [0,1] \to X$. In each small coordinate neighbourhood of a point on the curve $\varphi_U\gamma$ is a curve in a disc in $\mathbf{C}$, and rotating the tangent vector γ' by $90°$ or $-90°$ defines an upper and lower half:



Identification on an overlapping neighbourhood is by a map which preserves angles, and in particular the sense – anticlockwise or clockwise – so the two upper halves agree on the overlap, and as we pass around the closed curve the strip is separated into two halves. But removing the central curve of a Möbius strip leaves it connected:

which gives a contradiction. □

From the classification of surfaces we see that a closed, connected Riemann surface is homeomorphic to a connected sum of tori.

3.2 Meromorphic functions

Recall that on a closed (i.e. compact) surface X, any continuous real function achieves its maximum at some point x. Let X be a Riemann surface and f a holomorphic function, then $|f|$ is continuous, so assume it has its maximum at x. Since $f\varphi_U^{-1}$ is a holomorphic function on an open set in $\mathbf{C}$ containing $\varphi_U(x)$, and has its maximum modulus there, the maximum modulus principle says that f must be a constant c in a neighbourhood of x. If X is connected, it follows that $f = c$ everywhere.

Though there are no holomorphic functions, there do exist meromorphic functions:

Definition 10 *A meromorphic function f on a Riemann surface X is a holomorphic map to the Riemann sphere $S = \mathbf{C} \cup \{\infty\}$.*

This means that if we remove $f^{-1}(\infty)$, then f is just a holomorphic function F with values in $\mathbf{C}$. If $f(x) = \infty$, and U is a coordinate neighbourhood of x, then using the coordinate z', $f\varphi_U^{-1}$ is holomorphic. But $\tilde{z} = 1/z$ if $z \neq 0$ which means that $(F \circ \varphi_U^{-1})^{-1}$ is holomorphic. Since it also vanishes,

$$F \circ \varphi_U^{-1} = \frac{a_0}{z^m} + \ldots$$

which is usually what we mean by a meromorphic function.

Example: A rational function

$$f(z) = \frac{p(z)}{q(z)}$$

where p and q are polynomials is a meromorphic function on the Riemann sphere S.

The definition above is a geometrical one. Algebraically it is clear that the sum and product of meromorphic functions is meromophic – they form a field.

Here is an example of a meromorphic function on the torus in Example 2.

Define

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

where the sum is over all non-zero $\omega = m\omega_1 + n\omega_2$. Since for $2|z| < |\omega|$

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| \leq 10 \frac{|z|}{|\omega|^3}$$

this converges uniformly on compact sets so long as

$$\sum_{\omega \neq 0} \frac{1}{|\omega|^3} < \infty.$$

But $m\omega_1 + n\omega_2$ is never zero if m, n are real so we have an estimate

$$|m\omega_1 + n\omega_2| \geq k\sqrt{m^2 + n^2}$$

so by the integral test we have convergence. Because the sum is essentially over all equivalence classes

$$\wp(z + m\omega_1 + n\omega_2) = \wp(z)$$

so that this is a meromorphic function on the surface X. It is called the Weierstrass P-function.

It is a quite deep result that any closed Riemann surface has meromorphic functions. Let us consider them in more detail. So let

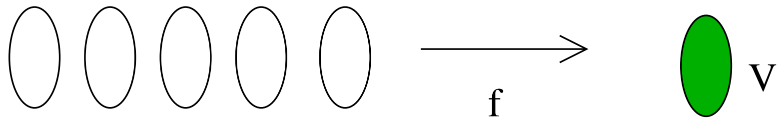
$$f : X \to S$$

be a meromorphic function. If the inverse image of $a \in S$ is infinite, then it has a limit point x by compactness of X. In a holomorphic coordinate around x with $z(x) = 0$, f is defined by a holomorphic function $F = f\varphi_U^{-1}$ with a sequence of points $z_n \to 0$ for which $F(z_n) - a = 0$. But the zeros of a holomorphic function are isolated, so we deduce that $f^{-1}(a)$ is a finite set. By a similar argument the points at which the derivative F' vanishes are finite in number (check that this condition is independent of the holomorphic coordinate). The points of X at which $F' = 0$ are called *ramification points*.

Now recall another result from complex analysis: if a holomorphic function f has a zero of order n at $z = 0$, then for $\epsilon > 0$ sufficiently small, there is $\delta > 0$ such that for all a with $0 < |a| < \delta$, the equation $f(z) = a$ has exactly n roots in the disc $|z| < \epsilon$.

This result has two consequences. The first is that if $F'(x) \neq 0$, then f maps a neighbourhood U_x of $x \in X$ homeomorphically to a neighbourhood V_x of $f(x) \in S$.

Define V to be the intersection of the V_x as x runs over the finite set of points such that $f(x) = a$, then $f^{-1}V$ consists of a finite number d of open sets, each mapped homeomorphically onto V by f:



The second is that if $F' = 0$, we have

$$F(z) = z^n(a_0 + a_1 z + \dots)$$

for some n and F has a zero of order n at 0, where $z(x) = 0$. In that case there is a neighbourhood U of x and V of a such that $f(U) = V$, and the inverse image of $y \neq x \in V$ consists of n distinct points, but $f^{-1}(a) = x$. In fact, since $a_0 \neq 0$, we can expand

$$(a_0 + a_1 z + \dots)^{1/n} = a_0^{1/n}(1 + b_1 z + \dots)$$

in a power series and use a new coordinate

$$w = a_0^{1/n} z(1 + b_1 z + \dots)$$

so that the map f is locally

$$w \mapsto w^n.$$

There are then two types of neighbourhoods of points: at an ordinary point the map looks like $w \mapsto w$ and at a ramification point like $w \mapsto w^n$.

Removing the finite number of images under f of ramification points we get a sphere minus a finite number of points. This is connected. The number of points in the inverse image of a point in this punctured sphere is integer-valued and continuous, hence constant. It is called the *degree* d of the meromorphic function f.

With this we can determine the Euler characteristic of the Riemann surface S from the meromorphic function:

Theorem 3.2 *(Riemann-Hurwitz) Let $f : X \to S$ be a meromorphic function of degree d on a closed connected Riemann surface X, and suppose it has ramification points $x_1, \dots, x_n$ where the local form of $f(x) - f(x_k)$ is a holomorphic function with a zero of multiplicity m_k. Then*

$$\chi(X) = 2d - \sum_{k=1}^{n}(m_k - 1)$$

Proof: The idea is to take a triangulation of the sphere S such that the image of the ramification points are vertices. This is straighforward. Now take a finite subcovering of S by open sets of the form V above where the map f is either a homeomorphism or of the form $z \mapsto z^m$. Subdivide the triangulation into smaller triangles such that each one is contained in one of the sets V. Then the inverse images of the vertices and edges of S form the vertices and edges of a triangulation of X.

If the triangulation of S has V vertices, E edges and F faces, then clearly the triangulation of X has dE edges and dF faces. It has fewer vertices, though — in a neighbourhood where f is of the form $w \mapsto w^m$ the origin is a single vertex instead of m of them. For each ramification point of order m_k we therefore have one vertex instead of m_k. The count of vertices is therefore

$$dV - \sum_{k=1}^{n}(m_k - 1).$$

Thus

$$\chi(X) = d(V - E + F) - \sum_{k=1}^{n}(m_k - 1) = 2d - \sum_{k=1}^{n}(m_k - 1)$$

using $\chi(S) = 2$. □

Clearly the argument works just the same for a holomorphic map $f : X \to Y$ and then

$$\chi(X) = d\chi(Y) - \sum_{k=1}^{n}(m_k - 1).$$

As an example, consider the Weierstrass P-function $\wp : T \to S$:

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$

This has degree 2 since $\wp(z) = \infty$ only at $z = 0$ and there it has multiplicity 2. Each $m_k \leq d = 2$, so the only possible value at the ramification points here is $m_k = 2$. The Riemann-Hurwitz formula gives:

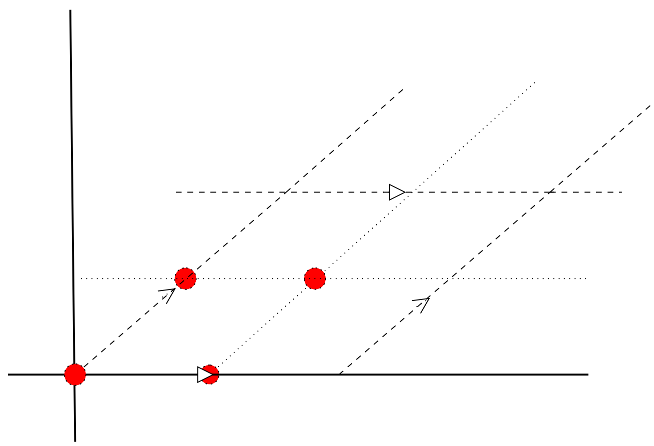
$$0 = 4 - n$$

so there must be exactly 4 ramification points. In fact we can see them directly, because $\wp(z)$ is an even function, so the derivative vanishes if $-z = z$. Of course at $z = 0$, $\wp(z) = \infty$ so we should use the other coordinate on S: $1/\wp$ has a zero of multiplicity 2 at $z = 0$. To find the other points recall that $\wp$ is doubly periodic so $\wp'$ vanishes where

$$z = -z + m\omega_1 + n\omega_2$$

for some integers m, n, and these are the four points

$$0, \omega_1/2, \omega_2/2, (\omega_1 + \omega_2)/2 :$$



3.3 Multi-valued functions

The Riemann-Hurwitz formula is useful for determining the Euler characteristic of a Riemann surface defined in terms of a multi-valued function, like

$$g(z) = z^{1/n}.$$

We look for a closed surface on which z and $g(z)$ are meromorphic functions. The example above is easy: if $w = z^{1/n}$ then $w^n = z$, and using the coordinate $z' = 1/z$ on a neighbourhood of ∞ on the Riemann sphere S, if $w' = 1/w$ then $w'^n = z'$.

Thus w and w' are standard coordinates on S, and $g(z)$ is the identity map $S \to S$. The function $z = w^n$ is then a meromorphic function f of degree n on S. It has two ramification points of order n at $w = 0$ and $w = \infty$, so the Riemann-Hurwitz formula is verified:

$$2 = \chi(S) = 2n - 2(n-1).$$

The most general case is that of a complex algebraic curve $f(z, w) = 0$. This is a polynomial in w with coefficients functions of z, so its "solution" is a multivalued function of z. We shall deal with a simpler but still important case $w^2 = p(z)$ where p is a polynomial of degree n in z with n distinct roots. We are looking then for a Riemann surface on which

$$\sqrt{p(z)}$$

can be interpreted as a meromorphic function.

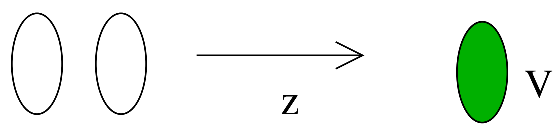
We proceed to define coordinate neighbourhoods on each of which w is a holomorphic function. First, if $p(a) \neq 0$ then

$$p(z) = p(a)(1 + a_1(z-a) + \ldots + a_n(z-a)^n).$$

For each choice of $\sqrt{p(a)}$ we have w expressed as a power series in z in a neigbourhood of a:

$$w = \sqrt{p(a)}(1 + a_1(z-a) + \ldots + a_n(z-a)^n)^{1/2} = 1 + a_1 z/2 + \ldots.$$

So we can take z as a coordinate on each of two open sets, and w is holomorphic here.



If $p(a) = 0$, then since p has distinct roots,

$$p(z) = (z-a)(b_0 + b_1(z-a) + \ldots)$$

where $b_0 \neq 0$. Put $u^2 = (z-a)$ and $p(z) = u^2(b_0 + b_1 u^2 + \ldots)$ and so, choosing $\sqrt{b_0}$, w has a power series expansion in u:

$$w = u\sqrt{b_0}(1 + b_1 u^2/b_0 + \ldots).$$

(The other choose of $\sqrt{b_0}$ is equivalent to taking the local coordinate $-u$.) This gives an open disc, with u as coordinate, on which w is holomorphic.

For $z = \infty$ we note that

$$\frac{w^2}{z^n} = a_n + \frac{a_{n-1}}{z} + \ldots$$

so if $n = 2m$,

$$\left(\frac{w}{z^m}\right)^2 = a_n + \frac{a_{n-1}}{z} + \ldots$$

and since $a_n \neq 0$, putting $w' = 1/w$ and $z' = 1/z$ we get

$$w' = a_n^{-1/2} z'^m (1 + a_{n-1} z'/a_n + \ldots)^{-1/2}$$

which is a holomorphic function. If $n = 2m + 1$, we need a coordinate $u^2 = z'$ as above.

The coordinate neighbourhoods defined above give the set of solutions to $w^2 = p(z)$ together with points at infinity the structure of a compact Riemann surface X such that

- z is a meromorphic function of degree 2 on X
- w is a meromorphic function of degree n on X
- the ramification points of z are at the points $(z = a, w = 0)$ where a is a root of $p(z)$, and if n is odd, at $(z = \infty, w = \infty)$

The Riemann-Hurwitz formula now gives

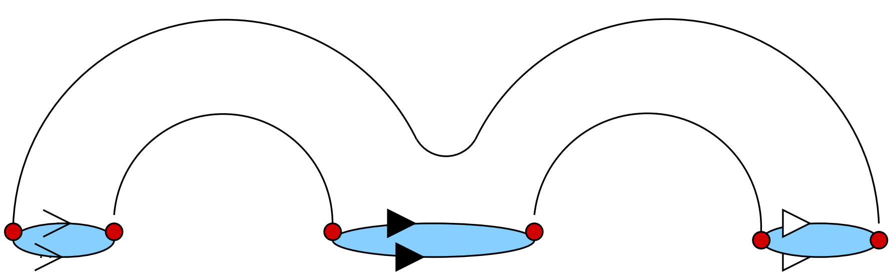
$$\chi(X) = 2 \times 2 - n = 4 - n$$

if n is even and

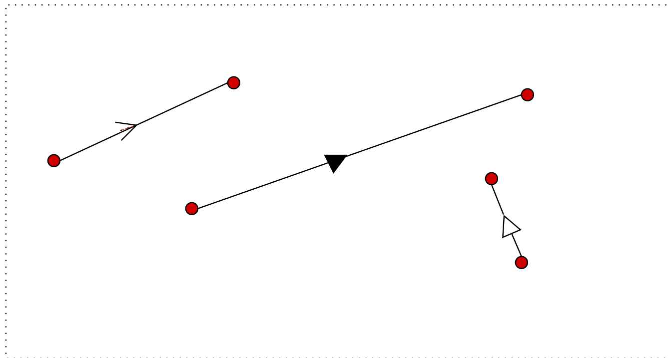
$$\chi(X) = 4 - (n + 1) = 3 - n$$

if n is odd.

This type of Riemann surface is called *hyperelliptic*. Since the two values of $w = \sqrt{p(z)}$ only differ by a sign, we can think of $(w, z) \mapsto (-w, z)$ as being a holomorphic homeomorphism from X to X, and then z is a coordinate on the space of orbits. Topologically we can cut the surface in two – an "upper" and "lower" half – and identify on the points on the boundary to get a sphere:



It is common also to view this downstairs on the Riemann sphere and insert cuts between pairs of zeros of the polynomial $p(z)$:



As an example, consider again the P-function $\wp(z)$, thought of as a degree 2 map $\wp : T \to S$. It has 4 ramification points, whose images are ∞ and the three finite points e_1, e_2, e_3 where

$$e_1 = \wp(\omega_1/2), \quad e_2 = \wp(\omega_2/2), \quad e_3 = \wp((\omega_1+\omega_2)/2).$$

So its derivative $\wp'(z)$ vanishes only at three points, each with multiplicity 1. At each of these points $\wp$ has the local form

$$\wp(z) = e_1 + (z - \omega_1/2)^2(a_0 + \ldots)$$

and so

$$\frac{1}{\wp'(z)^2}(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

is a well-defined holomorphic function on T away from $z = 0$. But $\wp(z) \sim z^{-2}$ near $z = 0$, and so $\wp'(z) \sim -2z^{-3}$ so this function is finite at $z = 0$ with value $1/4$. By the maximum argument, since T is compact,the function is a constant, namely $1/4$.

Thus the meromorphic function $\wp'(z)$ on T can also be considered as

$$2\sqrt{(u-e_1)(u-e_2)(u-e_3)}$$

setting $u = \wp(z)$.

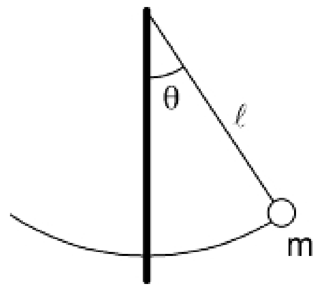
Note that, substituting $u = \wp(z)$, we have

$$\frac{du}{2\sqrt{(u-e_1)(u-e_2)(u-e_3)}} = dz.$$

By changing variables with a Möbius transformation of the form $u \mapsto (au+b)/(cu+d)$ any integrand

$$\frac{du}{\sqrt{p(u)}}$$

can be brought into this form if p is of degree 3 or 4. This can be very useful, for example in the equation for a pendulum:



$$\theta'' = -(g/\ell)\sin\theta$$

which integrates once to

$$\theta'^2 = 2(g/\ell)\cos\theta + c.$$

Substituting $v = e^{i\theta}$ we get

$$v' = i\sqrt{2(g/\ell)(v^3 + v) + cv^2}.$$

So time becomes (the real part of) the parameter z on $\mathbf{C}$. In the torus this is a circle, so (no surprise here!) the solutions to the pendulum equation are periodic.