

	<h1>How Euler Did It</h1> <p>by Ed Sandifer</p>	
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## Euler's Greatest Hits

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People love Top Ten lists. Johnny Carson and Dick Clark made their careers on them. The culture of the mathematical community changes more slowly than Carson's current events or Clark's pop music, so we have fewer reasons and opportunities to make lists.

There have been a couple of examples, though. David Wells did a survey for *Mathematical Intelligencer* in 1988 that put  $e^{\pi i} + 1 = 0$  at the top of the list of the most beautiful theorems in mathematics. In 2004, *Physics World* put the same formula second in their list of greatest equations, behind the Maxwell equations. *Physics World* also put  $1 + 1 = 2$  eighth on their list, praising its elegance and simplicity.

We recently had a chance to do an informal survey on Euler. Rob Bradley and I organized a Short Course at the Joint Mathematics Meetings in New Orleans in January 2007. During the course, after Ron Calinger's account of Euler's life and times, but before any specific mathematical content, we gave the participants in the Short Course a ballot listing 30 candidates as "Euler's greatest theorems." We asked for additional nominations from the floor, but there weren't any. Then we asked the participants to mark the theorems they were sure should be ranked among Euler's Top Ten, and also the ones they thought should be in the Top Three.

When we counted the ballots, we counted Top Three votes as three votes each, and the other Top Ten votes as one vote each. There were 35 ballots. The results are given below (with number of votes in parentheses.) Despite the flaws in the research methodology, these people were well informed in mathematics and interested in Euler, so until somebody else does another study, we can declare this to be the

### Official List of Euler's Top Ten Theorems.

1. (26) Basel problem:  $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$
2. (25) Polyhedral formula:  $V - E + F = 2$
3. (23)  $e^{\pi i} = -1$
4. (16) Königsberg bridge problem and the Knight's tour

5. (14) Euler product formula:  $\prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^n}} = \sum_{k=1}^{\infty} \frac{1}{k^n}$
6. (12) Euler-Lagrange necessary condition: If a function  $y$  makes  $J = \int_a^b f(x, y, y') dx$  a maximum or a minimum, then  $\frac{\partial f}{\partial y} - \frac{d}{dt} \left( \frac{\partial f}{\partial y'} \right) = 0$ .
7. (11) Density of primes:  $\sum_{p \text{ prime}} \frac{1}{p}$  diverges
8. (10) Generating functions and partition numbers
- 9T. (9) Euler-Fermat theorem:  $a^{J(n)} \equiv 1 \pmod{n}$
- 9T. (9) Gamma function

My own Top Ten list had seven of the same ten theorems in the Top Ten, and three of the Top Five, overlaps of 70% and 60% respectively. Among the Top Five, I left off  $e^{pi} = -1$  and Königsberg, and instead included the Euler-Fermat theorem and the Euler-Lagrange necessary condition.

I'd also left Königsberg out of my Top Ten, as well as the Density of primes and the Gamma function. Instead, I'd included the differential equations of fluid flow, the differential equations of the motions of solid bodies, and the Euler line theorem.

Reasonable people can disagree, and people tended to vote for the theorems that were most important in their own mathematical backgrounds. Number theorists voted for Fermat's Last Theorem in the cases  $n = 3$  and  $n = 4$ . Robin Wilson, author of a book on the history of graph theory, [BLW] voted for the Königsberg bridges, and Fred Rickey, professor of mathematics at the US Military Academy at West Point, had voted for the theorems in Euler's *Artillerie*.

There were apparently no differential geometers in the audience, as nobody at all voted for the orthogonality of principal curvatures on a surface. Typesetters seem scarce as well, since the typographically beautiful, though not very deep differential  $de^{e^{e^{e^x}}} = de^{e^{e^{e^x}}} e^{e^{e^x}} e^{e^x} e^x dx$  got only four votes. There may have been a few 19<sup>th</sup> century geometers, but they split their votes between the Nine-point circle theorem and the Euler line, which came in 15<sup>th</sup> and 23<sup>rd</sup> respectively. Mathematicians do have a common body of knowledge, so there was enough consensus that the top three theorems had a substantial lead over the rest, and all three surveys agreed that  $e^{pi} = -1$  belongs in the top three, even if they couldn't agree whether that's the better way to write it. Some prefer  $e^{pi} + 1 = 0$ .

Even beyond  $e^{pi} + 1 = 0$ , our Official List is consistent with the lists from the *Intelligencer* and from *Physics World*. The *Intelligencer* had the following Euler and Euler-related theorems in their Top Ten:

1.  $e^{pi} = -1$
2.  $V - E + F = 2$
3. Infinitude of primes. (They meant Euclid's proof, but  $\sum_{p \text{ prime}} \frac{1}{p}$  can't diverge unless there are infinitely many primes.)

$$5. \quad z(2) = \frac{p^2}{6}$$

10. If  $p$  is a prime number of the form  $4n + 1$ , then  $p$  can be written as the sum of two squares in exactly one way.

*Physics World* listed another Euler-related equation. In ninth position, they put the Principle of Least Action,  $dS = 0$ , the physical and metaphysical principle that anything that happens occurs in such a way as to maximize or minimize some quantity, such as energy, momentum, time or distance. This is related to the Euler-Lagrange necessary condition because, as we have written the formulas, the quantity  $J$  is maximized or minimized when  $dJ = 0$ .

Each month, this column mostly describes the details of Euler's mathematics, finding clever, interesting and beautiful ways Euler did things. We try to put those details into the contexts of Euler's times and his other work. There are enough columns (this one is # 40 in the series) that the details combine to give a fairly extensive piece of the picture of Euler's work. Still, we've made no particular effort to give an overall picture of his work, to identify which of that work is his best, or to answer the questions "Why was Euler great?" and "Why is he famous?" Now, armed with the Official List, we can look at the columns so far and gage how much of the Big Picture we have looked at by comparing column topics with the items on the List. There is, of course, a huge bias towards mathematics in this view, and it woefully neglects Euler's work in other fields. Only about 40% of his books and papers are mathematics. The other 60% range over mechanics, optics, astronomy, magnetism, acoustics, philosophy and half a dozen other topics. Anyway, how are we doing in the Top Ten?

### 1. The Basel Problem

Since William Dunham [D] gives such a beautiful account of Euler's famous infinite-product solution to the Basel Problem, we've not tried to write a column about it. In 1741, though, Euler gave a completely different solution, and that was the subject of the March 2004 column, "Basel Problem with Integrals." Also, we described Euler's numerical solution, finding the sum of the reciprocals of the squares to six decimal places, in "Estimating the Basel Problem," December 2003.

### 2. $V - E + F = 2$

We gave a detailed analysis of the theorem and the flaws in Euler's proof in two consecutive columns, June and July 2004, " $V - E + F = 2$ ," parts 1 and 2.

### 3. $e^{pi} = -1$

I have a problem with this. Though I agree that it is a beautiful and important result, I am not convinced that we are right to attribute it to Euler.

First, I've never seen Euler state the fact in this way. However, in his very first letter to Christian Goldbach in October 1729, the 22-year old Euler wrote that a certain sum

"is equal to this:  $\frac{1}{2}\sqrt{\sqrt{-1} \cdot \ln(-1)}$ , which is equal to the side of the square equal to the circle with diameter equal to 1."

We can decode this. A circle with diameter 1 has radius  $\frac{1}{2}$ , so its area is  $\pi/4$ . The square root of that is  $\frac{1}{2}\sqrt{\pi}$ . So, Euler is claiming that  $\sqrt{\sqrt{-1} \cdot \ln(-1)} = \sqrt{\pi}$ . Hitting this with a little bit of algebra gives us

$$\begin{aligned}i \ln(-1) &= \pi \\ \ln(-1) &= -\pi i \\ -1 &= e^{-\pi i}\end{aligned}$$

Now, take reciprocals of both sides to get

$$e^{\pi i} = -1.$$

So, Euler knew something easily equivalent to the formula as early as 1729. It's hard to believe he'd figured it out for himself, so I think he learned it from Johann Bernoulli.

Moreover, it's pretty clear that Cotes knew the generalization,  $e^{iq} = \cos q + i \sin q$  before Euler came on the scene.

It's a beautiful result, but until I get this attribution issue straightened out, there probably won't be a column on  $e^{\pi i} = -1$ .

#### **4. Königsberg bridges and the Knight's tour**

There are many descriptions of the Königsberg bridge problem. My favorite is [BLW]. With such fine expositions easily available, it doesn't make sense to devote a column to Königsberg. On the other hand, we described Euler's work on the Knight's tour in April 2006.

#### **5. Euler product formula**

We described Euler's proof of the product sum formula in March 2006, "Infinitely Many Primes," though the point of that column was more closely related to item 7 on our Top Ten list, the Density of primes. We also mentioned it in the column "Formal Sums and Products," July 2006.

#### **6. Euler-Lagrange necessary condition**

The calculus of variations hasn't appeared yet in any of our columns. It is a beautiful and deep subject, but it has a reputation for being difficult for non-specialists to understand and appreciate. Perhaps we can find an opportunity in a future column,

#### **7. Density of primes**

The column from March 2006, "Infinitely Many Primes," describes most of Euler's work on this result in some detail.

#### **8. Generating functions and the partition problem**

We've written about this twice, first in "Roots by Recursion," June 2005, and then some different aspects of the same topic in "Philip Naudé's Problem," October 2005.

#### **9T. The Euler-Fermat theorem**

Back in November 2003, we devoted our very first column to Euler's proof of Fermat's Little Theorem. We haven't returned to the topic, though, so we haven't described either the Euler phi-function or the Euler-Fermat theorem.

## 9T. Gamma function

We haven't written a column about the gamma function, either. It would make a good column.

### Summary

Four of the theorems, Basel problem,  $e^{pi} = -1$ , Euler-Lagrange necessary condition and gamma function, have not been central topics of columns yet.

Two of the theorems, Königsberg/Knight's tour and Euler-Fermat theorem, have been about half covered.

Four of them,  $V - E + F = 2$ , Euler product formula, density of primes and generating functions/partitions have been well covered.

It looks like a lot of the picture is still missing. It will be a while before we run out of material for more columns.

References:

[D] Dunham, William, *Euler: The Master of Us All*, MAA, Washington, DC, 1999.

[BLW] Biggs, Norman, Keith Lloyd and Robin Wilson, *Graph Theory 1736-1936*, Oxford University Press, 1986.

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